Worcester County Mathematics League

Varsity Meet 3 - January 29, 2025

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 3 - January 29, 2025
Answer Key

Round 1 Similarity and Pythagorean Theorem

1. 24
2. 10
3. 25

Round 2 - Algebra I

1. $-9x^2 - 18x + 1$
2. $(16, 20)$
3. $20 - 14\sqrt{2}$

Round 3 - Functions

1. $x^2 - 4x + 5$
2. $\frac{7}{4}$ or $1\frac{3}{4}$ or 1.75
3. -2

Round 4 - Combinatorics

1. 72
2. 360
3. 12144

Round 5 - Analytic Geometry

1. -14
2. $3x + 4y = -7$ or $-3x - 4y = 7$
3. $y = -\frac{1}{20}x^2 + 5$

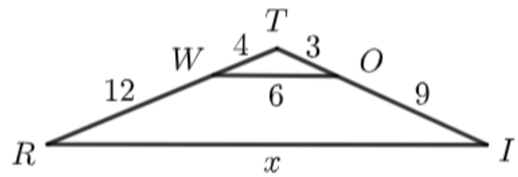
Team Round

1. 60
2. $5, 7, 9$ (any order, need all three)
3. $7k^2 - 16k + 10$
4. $\frac{4}{7}$
5. $(-3, 1)$
6. $\frac{105 + 56\sqrt{3}}{15}$
7. 1
8. 480
9. $\frac{8}{9}$ or $0.\overline{8}$

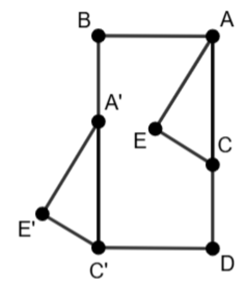
Worcester County Mathematics League
Varsity Meet 3 - January 29, 2025
Round 1 - Similarity and Pythagorean Theorem

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

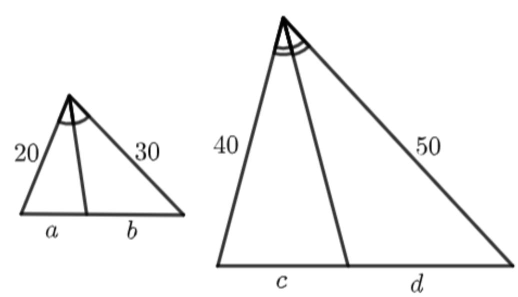
1. Given $\triangle TRI$ shown at right with $TW = 4$, $TO = 3$, $WO = 6$, $WR = 12$, and $OI = 9$, find x (RI).



2. Given rectangle $ABC'D$ shown at right. where $BC' = 15$, $AB = 8$, $AC = 9$, $A'E' = 8$, and $\triangle ACE \cong \triangle A'C'E'$; find EE' (the length of $\overline{EE'}$, not drawn).



3. Given the segment lengths as labeled on the two triangles at right, where $a + b + c + d = 70$ and $c = 2a$, find d.



ANSWERS

(1 pt) 1. $x =$ __________________

(2 pts) 2. $EE' =$ __________________

(3 pts) 3. $d =$ __________________

Bancroft, Tahanto, Tahanto

Worcester County Mathematics League
Varsity Meet 3 - January 29, 2025
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. What polynomial must be added to $9x + 4$ to result in $-9x^2 - 9x + 5$? Express your answer in standard form, that is, $ax^2 + bx + c$.

2. The fraction $\frac{m}{n}$ is equal to $\frac{4}{5}$. The fraction $\frac{m+9}{n}$ is equal to $\frac{5}{4}$. Find the ordered pair (m, n).

3. Solve the following equation for x:

$$x^{\frac{1}{3}} - 8^{\frac{1}{6}} = -\frac{4}{2 + 2\sqrt{2}}$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $(m, n) = ($________________$)$

(3 pts) 3. $x =$ ____________________

St. John's, Tantasqua, Doherty/QSC

Worcester County Mathematics League
Varsity Meet 3 - January 29, 2025
Round 3 - Functions

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Given functions $f(x) = x^2 + 1$ and $g(x) = x - 2$, find $f(g(x))$. Express your answer as a polynomial in standard form, that is, $ax^2 + bx + c$.

2. Given that $f(x) = \dfrac{3x+9}{5x-4}$, find $f^{-1}(3)$.

3. Given invertible functions $f(x) = x^3 + 6x$ and $g(x) = x^3 - 6x^2 + 18x - 20$, where $g^{-1}(x) = f^{-1}(x) - a$; find a.

ANSWERS

(1 pt) 1. $f(g(x)) =$ ________________

(2 pts) 2. $f^{-1}(3) =$ ________________

(3 pts) 3. $a =$ ________________

Doherty, Worc. Acad., QSC

Worcester County Mathematics League
Varsity Meet 3 - January 29, 2025
Round 4 - Combinatorics

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. How many ways can a captain and an assistant be chosen from a math team with nine members? A team member cannot be both captain and assistant.

2. There are three vowels (e,a,o) in the word “pentagon”. How many arrangements of the eight letters in the word “pentagon” begin with three vowels?

3. How many ways are there to place three distinct dividers (one red, one white, one blue) to divide a shelf of 27 identical books such that there are at least 2 books between each divider, and no divider is at either end of the books?

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Algonquin/QSC, St. Peter-Marian, Tantasqua

Worcester County Mathematics League
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Round 5 - Analytic Geometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Given that the points $(3, 6)$, $(2, 1)$, and $(-1, c)$ lie on a line (they are *collinear*), find c.

2. Given the circle with the equation:

$$(x-2)^2 + (y-3)^2 = 25$$

and a line l that is tangent to the circle at $(-1, -1)$, find the equation of l. Express your answer in standard form, that is, $Ax + By = C$, where A, B, C are integers that share no common factor.

3. Circles P and Q (with centers at distinct points P and Q) are tangent to both the x- and y-axes, and P and Q lie on the parabola with equation $x^2 = 20y - 100$. Let V be the vertex of this parabola. Find the equation of the parabola that contains V and the tangent points of circles P and Q with the $x-$axis. Express your equation in the form $y = ax^2 + b$.

ANSWERS

(1 pt) 1. $c =$ ________________

(2 pts) 2. ________________

(3 pts) 3. ________________

Auburn, Assabet Valley, Algonquin/QSC

Worcester County Mathematics League

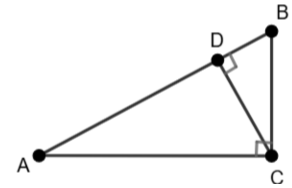
Varsity Meet 3 - January 29, 2025

Team Round

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Given right $\triangle ABC$ with altitude $\overline{CD}$ in the figure at right, where $AB = 17$, $A_1 =$ area of $\triangle ABC$, $A_2 =$ area of $\triangle CBD$, and $\frac{A_1}{A_2} = \frac{289}{64}$; find A_1.



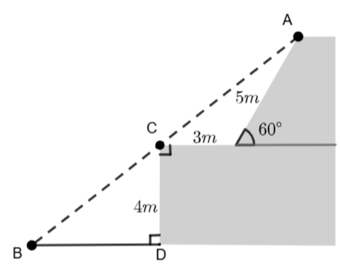
2. Find three consecutive odd integers, in increasing order, such that three times the middle integer is seven more than the sum of the first and last integers.

3. Let $f(x) = 7x^2 - 2x + 1$. Find $f(k-1)$. Express your answer as a polynomial in standard form, that is, $ak^2 + bk + c$.

4. Seven cards, exactly one of which is a joker, are randomly placed in a row. Ann is instructed to flip over three of the cards in succession. She is shown the adjacent card or cards to the flipped card after she flips it. Ann chooses a card flipping strategy that will guarantee that she knows the location of the joker before she flips the third card. Find the probability that Ann flips over the joker in her first or second try, given that she flips over the joker on her second flip if it is shown to her after her first flip. Express your answer as a fraction $\frac{m}{n}$.

5. Two circles are defined by the equations $(x+6)^2 + (y-1)^2 = 9$ and $(x+1)^2 + (y-1)^2 = 4$. Find the point or points of intersection of these circles, expressed as ordered pair(s) (x, y).

6. A building has the shaded profile shown at right, with a 4m vertical wall $\overline{CD}$, a 3m horizonal surface, and a 5m sloping wall that rises at an angle of $60°$ from the surface. A ladder is placed on the level and horizontal ground at point B and reaches exactly to point A at the corner of the sloping wall, touching the vertical wall at its corner C. What is the length of the ladder (AB)? Express your answer in simplest form as $\frac{m+p\sqrt{q}}{n}$.



7. Let a, b, and c be different integers between 1 and 9, inclusive. Find the largest possible value of
$$\frac{a+b+c}{abc}.$$

8. How many three-digit and four-digit numbers can be formed using the digits 1, 2, 3, 4, 5, and 6, if no digit is repeated within a single number?

9. Given points $A(0,0)$, $B(0,12)$, $C(12,4)$, $D(4,4)$, and $E(8,0)$, find $\dfrac{A_2 - A_1}{A_2}$, where A_2 is the area of $\triangle ABC$ and A_1 is the area of the overlap between $\triangle ABC$ and $\triangle ADE$.

Quaboag, Shepherd Hill, Bartlett, Monty Tech/QSC, Hopkinton, Tahanto, Hudson, Mass. Acad., Doherty

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. $A_1 =$ ________________

2. ________, ________, ________

3. ____________________

4. ____________________

5. ____________________ (answer must be ordered pair(s), e.g. "(6,-1)")

6. ____________________

7. ____________________

8. ____________________

9. $\frac{A_2 - A_1}{A_2} =$ ________________

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5. $(-3, 1)$
6. $\frac{105 + 56\sqrt{3}}{15}$
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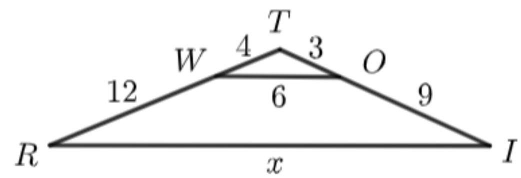
Round 1 - Similarity and Pythagorean Theorem

1. Given $\triangle TRI$ shown at right with $TW = 4$, $TO = 3$, $WO = 6$, $WR = 12$, and $OI = 9$, find x (RI).



Solution: The segment length RI can be found by noting that $\triangle TWO \sim \triangle TRI$ with corresponding sides $\overline{WO}$ and $\overline{RI}$. Then the scale factor from $\triangle TWO$ to $\triangle TRI$ can be applied to $WO = 6$ to find the length of corresponding side $\overline{RI}$.

Note that $\angle T$ is shared between $\triangle TWO$ and $\triangle TRI$. Also note that

$$\frac{TR}{TW} = \frac{4+12}{4} = \frac{16}{4} = 4,$$

and

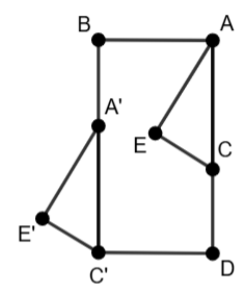
$$\frac{TI}{TO} = \frac{3+9}{3} = \frac{12}{3} = 4,$$

so that $\frac{TR}{TW} = \frac{TI}{TO}$. Therefore the sides of $\triangle TWO$ and $\triangle TRI$ that are adjacent to shared $\angle T$ are in proportion, and $\triangle TWO \sim \triangle TRI$ by SAS Similarity. The scale factor is the ratio of the lengths of corresponding sides: 4. Thus,

$$x = RI = 4 \cdot WO = 4 \cdot 6 = \boxed{24}$$

.

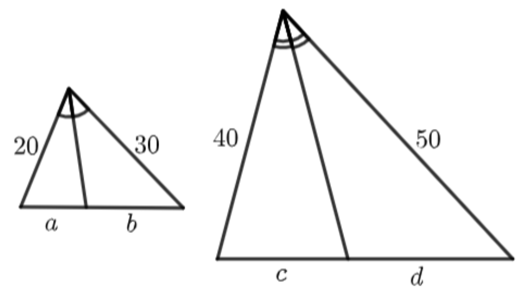
2. Given rectangle $ABC'D$ shown at right. where $BC' = 15$, $AB = 8$, $AC = 9$, $A'E' = 8$, and $\triangle ACE \cong \triangle A'C'E'$; find EE' (the length of $\overline{EE'}$, not drawn).



Solution: This problem is most easily solved by noting that $\triangle ACE$ can be translated, without rotation, so that it exactly overlaps $\triangle A'C'E'$. That is, the translation that maps A to A' will also map C to C' and E to E'. Therefore $EE' = CC' = AA'$.

Now consider $\triangle CDC'$. It is a right triangle with hypotenuse CC' because $ABC'D$ is a rectangle and $\angle D$ is a right angle. Note that $DC' = AB = 8$ and $CD = AD - AC = BC' - AC = 15 - 9 = 6$. Then by the Pythagorean Theorem $(CC')^2 = (CD)^2 + (DC')^2 = 6^2 + 8^2 = 36 + 64 = 100$ and $CC' = \sqrt{100} = 10$. To conclude, $EE' = CC' = \boxed{10}$.

3. Given the segment lengths as labeled on the two triangles at right, where $a+b+c+d=70$ and $c=2a$, find d.



Solution: This problem can be solved using the Angle Bisector Theorem, which states that the bisector of any angle of a triangle divides the opposite side of the triangle in proportion to the lengths of the sides adjacent to the angle. Write the proportions for the two triangles that are shown:

$$\frac{a}{b} = \frac{20}{30}$$
$$\frac{c}{d} = \frac{40}{50}$$

Now $a = \frac{20}{30} \cdot b = \frac{2}{3}b$, and $c = 2a = \frac{40}{50} \cdot d = \frac{4}{5}d$. From the given information, $a = \frac{1}{2}c = \frac{1}{2} \cdot \frac{4}{5}d = \frac{2}{5}d$. Then $b = \frac{3}{2}a = \frac{3}{2} \cdot \frac{2}{5}d = \frac{3}{5}d$.

Now we have expressions for a, b, and c, each in terms of d. Replace these three variables in the given sum equation and solve for d:

$$\begin{aligned} a+b+c+d &= 70 \\ \frac{2}{5}d + \frac{3}{5}d + \frac{4}{5}d + d &= 70 \\ \left(\frac{2}{5} + \frac{3}{5} + \frac{4}{5} + 1\right) d &= 70 \\ \frac{2+3+4+5}{5} d &= 70 \\ \frac{14}{5} d &= 70 \end{aligned}$$

and $d = \frac{5}{14} \cdot 70 = 5 \cdot \frac{70}{14} = 5 \cdot 5 = \boxed{25}$.

Round 2 - Algebra I

1. What polynomial must be added to $9x+4$ to result in $-9x^2-9x+5$? Express your answer in standard form, that is, ax^2+bx+c.

Solution: Let the desired polynomial be equal to ax^2+bx+c. Then

$$ax^2+bx+c+(9x+4)=-9x^2-9x+5.$$

Add $-(9x+4)=-9x-4$ to both sides of this equation, collect terms, and put the polynomial in standard form:

$$\begin{aligned} ax^2+bx+c &= -9x^2-9x+5+(-9)x-4 \\ &= -9x^2+(-9+(-9))x+5-4 \\ &= -9x^2+(-18)x+1 \\ &= \boxed{-9x^2-18x+1} \end{aligned}$$

2. The fraction $\frac{m}{n}$ is equal to $\frac{4}{5}$. The fraction $\frac{m+9}{n}$ is equal to $\frac{5}{4}$. Find the ordered pair (m,n).

Solution: Start by noting that $\frac{m+9}{n}=\frac{m}{n}+\frac{9}{n}$ so that:

$$\frac{5}{4}=\frac{m+9}{n}=\frac{m}{n}+\frac{9}{n}=\frac{4}{5}+\frac{9}{n}.$$

Therefore

$$\frac{9}{n}=\frac{5}{4}-\frac{4}{5}=\frac{5\cdot 5-4\cdot 4}{4\cdot 5}=\frac{25-16}{20}=\frac{9}{20}.$$

Thus, $n=20$. Then $\dfrac{m}{n}=\dfrac{m}{20}=\dfrac{4}{5}$, $m=20\cdot\dfrac{4}{5}=4\cdot\dfrac{20}{5}=4\cdot 4=16$, and $(m,n)=\boxed{(16,20)}$.

3. Solve the following equation for x:
$$x^{\frac{1}{3}} - 8^{\frac{1}{6}} = -\frac{4}{2+2\sqrt{2}}$$

Solution: Start by multiplying both the numerator and the denominator of the right hand side (RHS) of the equation by the conjugate of the denominator:

$$-\frac{4}{2+2\sqrt{2}} = -\left(\frac{4}{2+2\sqrt{2}}\right)\frac{2\sqrt{2}-2}{2\sqrt{2}-2} = -\frac{4(2\sqrt{2}-2)}{(2\sqrt{2}+2)(2\sqrt{2}-2)}$$

Now simplify the denominator using the identity $(a+b)(a-b) = a^2 - b^2$ and the law of exponents $(ab)^2 = a^2b^2$:

$$(2\sqrt{2}+2)(2\sqrt{2}-2) = (2\sqrt{2})^2 - 2^2 = 2^2(\sqrt{2})^2 - 4 = 4 \cdot 2 - 4 = 8 - 4 = 4.$$

Thus, the RHS is equal to $-\dfrac{4(2\sqrt{2}-2)}{4} = -(2\sqrt{2}-2) = 2 - 2\sqrt{2}$.

Next, note that $8 = 2^3$ and $8^{\frac{1}{6}} = (2^3)^{\frac{1}{6}} = 2^{\left(3\cdot\frac{1}{6}\right)} = 2^{\frac{1}{2}} = \sqrt{2}$, so that the left hand side (LHS) of the equation is equal to $x^{\frac{1}{3}} - \sqrt{2}$. Next, set this LHS expression equal to the simplified RHS and isolate $x^{\frac{1}{3}}$ on the LHS:

$$\begin{aligned} x^{\frac{1}{3}} - \sqrt{2} &= 2 - 2\sqrt{2} \\ x^{\frac{1}{3}} &= 2 - 2\sqrt{2} + \sqrt{2} \\ x^{\frac{1}{3}} &= 2 - \sqrt{2}. \end{aligned}$$

Now cube both sides of the equation, apply the binomial identity $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ to the RHS, and collect terms to simplify:

$$\begin{aligned} \left(x^{\frac{1}{3}}\right)^3 &= \left(2 - \sqrt{2}\right)^3 \\ x &= 2^3 - 3(2)^2\sqrt{2} + 3 \cdot 2(\sqrt{2})^2 - (\sqrt{2})^3 \\ &= 8 - 3 \cdot 4\sqrt{2} + 3 \cdot 2 \cdot 2 - 2\sqrt{2} \\ &= 8 + 12 - 12\sqrt{2} - 2\sqrt{2} = \boxed{20 - 14\sqrt{2}} \end{aligned}$$

Round 3 - Functions

1. Given functions $f(x) = x^2 + 1$ and $g(x) = x - 2$, find $f(g(x))$. Express your answer as a polynomial in standard form, that is, $ax^2 + bx + c$.

Solution: Begin with the definition $f(x)$, replacing x with $g(x) = x - 2$:

$$f(g(x)) = f(x-2) = (x-2)^2 + 1$$

Next, use the identity $(x-a)^2 = x^2 - 2ax + a^2$ to expand the square term. Then collect terms and simplify, writing the polynomial in standard form:

$$\begin{aligned} f(g(x)) &= x^2 - 2 \cdot 2x + 2^2 + 1 \\ &= x^2 - 4x + 4 + 1 \\ &= \boxed{x^2 - 4x + 5} \end{aligned}$$

2. Given that $f(x) = \dfrac{3x+9}{5x-4}$, find $f^{-1}(3)$.

Solution: Note that if $y = f(x)$ maps x to y, then $x = f^{-1}(y)$ maps y to x. Then set $f(x) = 3$ and solve for x to find $f^{-1}(3)$:

$$\begin{aligned} 3 &= f(x) \\ &= \frac{3x+9}{5x-4} \\ 3(5x-4) &= 3x+9 \\ 15x - 12 &= 3x + 9 \\ 15x - 3x &= 9 + 12 \\ 12x &= 21 \end{aligned}$$

and $x = \dfrac{21}{12} = \dfrac{7 \cdot 3}{4 \cdot 3} = \boxed{\dfrac{7}{4}}$.

3. Given invertible functions $f(x) = x^3 + 6x$ and $g(x) = x^3 - 6x^2 + 18x - 20$, where $g^{-1}(x) = f^{-1}(x) - a$; find a.

Solution: Note that the graph of g^{-1} is a vertical shift down by a units of the graph of f^{-1} since $g^{-1}(x) = f^{-1}(x) - a$. Then it can be shown that graph of g is a horizontal shift left by a units of the graph of f. That is, $g(x) = f(x + a)$.

Set $f(x + a) = g(x)$ and solve for a:

$$\begin{aligned} f(x+a) &= (x+a)^3 + 6(x+a) \\ &= x^3 + 3ax^2 + 3a^2x + a^3 + 6x + 6a \\ &= x^3 + 3ax^2 + (3a^2+6)x + a^3 + 6a \\ &= x^3 - 6x^2 + 18x - 20 = g(x) \end{aligned}$$

Compare the terms of the polynomials shown in the third and fourth lines above. Find a by setting the x^2 coefficients equal to each other: $3a = -6$ so that $a = \frac{-6}{3} = \boxed{-2}$.

This answer can be checked by setting $a = -2$ in the third polynomial and comparing the x and constant coefficients to the fourth polynomial. Thus, $3a^2 + 6 = 3(-2)^2 + 6 = 3 \cdot 4 + 6 = 12 + 6 = 18$, and $a^3 + 6a = (-2)^3 + 6(-2) = -8 + -12 = -20$, confirming that $f(x - 2) = g(x)$.

Finally, the assertion (that g's graph must be a horizontal shift left of f's graph if g^{-1}'s graph is a vertical shift down of f^{-1}'s graph) can be shown as follows.

Note that $g\left(g^{-1}(x)\right) = x$ (because g is the inverse of g^{-1}) and likewise $f\left(f^{-1}(x)\right) = x$, so that $g\left(g^{-1}(x)\right) = f\left(f^{-1}(x)\right)$ *for all values of* x. Then

$$\begin{aligned} g\left(g^{-1}(x)\right) &= g\left(f^{-1}(x) - a\right) \\ &= f\left(f^{-1}(x)\right) \end{aligned}$$

Now let $y = f^{-1}(x)$ so that $g(y - a) = f(y)$. Let $y - a = x$, or $y = x + a$. Then $g(x) = f(x + a)$, Q. E. D.

Round 4 - Combinatorics

1. How many ways can a captain and an assistant be chosen from a math team with nine members? A team member cannot be both captain and assistant.

Solution: There are nine choices for the captain. Once the captain is chosen, there are eight choices for the assistant. The number of captain/assistant choices is therefore $9 \cdot 8 = \boxed{72}$. Note that this calculation is ${}_9P_2$, a permutation of two chosen from nine.

2. There are three vowels (e,a,o) in the word "pentagon". How many arrangements of the eight letters in the word "pentagon" begin with three vowels?

Solution: There are ${}_3P_3 = 3 \cdot 2 \cdot 1 = 6$ ways to order the first three letters.

Now there are 5 remaining letters (the consonants p,n,t,g,n) that can be ordered $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways, except that two of the letters are n's. This number (120) overcounts the number of arrangements by a factor of two, as can be seen by designating the two n's as n_1 and n_2. Then for any arrangement, there is a paired arrangement with the positions of n_1 and n_2 swapped; both arrangements are included in the 120 count, but they are the same arrangement. Thus there $120 \div 2 = 60$ ways to order the last five letters.

Finally, the arrangements of the first three and last five letters are independent. Therefore the total number of eight letter arrangements is the product of the two counts: $6 \cdot 60 = \boxed{360}$.

3. How many ways are there to place three distinct dividers (one red, one white, one blue) to divide a shelf of 27 identical books such that there are at least 2 books between each divider, and no divider is at either end of the books?

Solution: The easiest way to arrive at an answer for this problem is to treat it as a modified "stars and bars" problem. The unmodified "stars and bars" problem asks how many ways there are to divide n stars into $k+1$ distinct bins, with k bars to divide them. The stars are arranged in a line and the number of stars in each bin are determined by the location of k bars. There are $n+k$ locations for each one of the k bars, so there are ${}_{n+k}C_k$, or $n+k$ choose k ways to choose positions for the bars, and thus ${}_{n+k}C_k$ ways to divide the n bars into $k+1$ bins.

Note that this method allows two bars to be adjacent to each other, that is, zero stars between them. Also, there may be zero stars to the left of the leftmost bar and there may be zero stars to the right of the rightmost bar.

In this problem, the books are "stars" and the dividers are "bars". The first constraint requires at least two books between the first and second dividers, and at least two more books between the second and third dividers. These four books are excluded from the "stars and bars" count, since their location is fixed for all divider placements. The second constraint of the problem requires that there must be at least one of the 27 books to the left of the leftmost divider, and at least one of the 27 books to the right of the rightmost divider. These two books are also excluded from the "stars and bars" count, for a total of $4+2=6$ excluded books. Then $n = 27-6 = 21$, and $k=3$.

If the dividers were identical, the placement count would be ${}_{21+3}C_3 = {}_{24}C_3 = \frac{24\cdot 23\cdot 22}{3\cdot 2\cdot 1}$. However, the dividers are distinct and their placement order affects the count. There are $3\cdot 2\cdot 1$ ways to order the dividers for each division of books. The final count is the product of the divider orderings and the number of divisions, or $3\cdot 2\cdot 1\cdot \frac{24\cdot 23\cdot 22}{3\cdot 2\cdot 1} = 24\cdot 23\cdot 22 = \boxed{12144}$.

As a final note, here is one method to perform the required multiplication:

$$\begin{aligned}
22\cdot 23\cdot 24 &= 12\cdot 2\cdot 23\cdot 22\\
&= 12\cdot (23\cdot 44)\\
&= 12\cdot ((20+3)(40+4))\\
&= 12\cdot (800+120+80+12)\\
&= 12\cdot (1000+12)\\
&= 12000+12^2\\
&= 12000+144 = 12144
\end{aligned}$$

Round 5 - Analytic Geometry

1. Given that the points $(3, 6)$, $(2, 1)$, and $(-1, c)$ lie on a line (they are *collinear*), find c.

Solution: Begin by finding the slope m of the line containing the first two points, with the goal of finding that line's equation. Set $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (2, 1)$ in the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-6}{2-3} = \frac{-5}{-1} = 5$$

To complete the slope-intercept form of the equation, plug $m = 5$ and $(x, y) = (2, 1)$ into $y = mx+b$:

$$1 = 5(2) + b = 10 + b$$

and $b + 10 = 1$, or $b = 1 - 10 = -9$. Thus, the equation of the line is $y = 5x - 9$. Finally, plug $(x, y) = (-1, c)$ into this equation and solve for c:

$$c = 5(-1) - 9 = -5 - 9 = \boxed{-14}$$

2. Given the circle with the equation:

$$(x-2)^2 + (y-3)^2 = 25$$

and a line l that is tangent to the circle at $(-1, -1)$, find the equation of l. Express your answer in standard form, that is, $Ax + By = C$, where A, B, C are integers that share no common factor.

Solution: The line l is perpendicular to the radius drawn to the point of tangency, that is, $(-1, -1)$. The center of the circle is $(2, 3)$ because the circle's equation is in standard form:

$$(x-h)^2 + (y-k)^2 = r^2$$

where the circle's center is (h, k) and its radius is r. Then the slope of the radius is found by the slope equation for points $(-1, -1)$ and $(2, 3)$:

$$m = \frac{3-(-1)}{2-(-1)} = \frac{3+1}{2+1} = \frac{4}{3}$$

The slope of l must be the negative reciprocal of m, or $-\frac{3}{4}$. The equation of the line l can be found by plugging this slope value and the point $(-1, -1)$ into the point-slope equation:

$$\begin{aligned} -\frac{3}{4} &= \frac{y-(-1)}{x-(-1)} = \frac{y+1}{x+1} \\ -3(x+1) &= 4(y+1) \\ -3x - 3 &= 4y + 4 \end{aligned}$$

Put this equation into standard form by adding $3x-4$ to both sides, and the answer is $\boxed{3x + 4y = -7}$.

3. Circles P and Q (with centers at distinct points P and Q) are tangent to both the x- and y-axes, and P and Q lie on the parabola with equation $x^2 = 20y - 100$. Let V be the vertex of this parabola. Find the equation of the parabola that contains V and the tangent points of circles P and Q with the $x-$axis. Express your equation in the form $y = ax^2 + b$.

Solution: First, note that the axis of symmetry for the parabola is $x = 0$ (the y-axis). Setting $x = 0$ and solving for y gives the y-coordinate of the vertex: $0 = 20y - 100$, $20y = 100$, and $y = 5$. The vertex is $V(0, 5)$.

Next, note that for circles P and Q to be tangent to both the x- and y-axes, their centers P and Q must be equidistant from the two axes, so that they lie either on the line $y = x$ or the line $y = -x$. Find P and Q by substituting $y = \pm x$ into the equation for the parabola and solving for x:

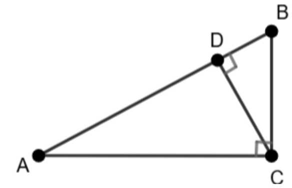
$$\begin{aligned} x^2 &= 20(\pm x) - 100 \\ x^2 \pm 20x + 100 &= 0 \\ (x \pm 10)^2 &= 0 \\ x &= \pm 10 \end{aligned}$$

where the quadratic was factored into a perfect square in the third line. Plug $x^2 = (\pm 10)^2 = 100$ into the parabola equation and $100 = 20y - 100$, so $20y = 200$ and $y = 10$. Thus $P(-10, 10)$ and $Q(10, 10)$ are the circle centers. Then the circles are tangent to the x-axis at points $T_P(-10, 0)$ and $T_Q(10, 0)$.

Clearly the axis of symmetry for the parabola containing T_P and T_Q is the y-axis, that is, $x = 0$, so $V(0, 5)$ is the vertex for this second parabola, as well as the first. Substitute $(x, y) = (0, 5)$ into $y = ax^2 + b$ and $5 = a(0)^2 + b = b$. Substitute $(x, y) = (10, 0)$ and $b = 5$ into $y = ax^2 + b$ and $0 = a(10)^2 + 5$, $100a = -5$, or $a = -\frac{5}{100} = -\frac{1}{20}$. Finally, the equation of the parabola is $\boxed{y = -\dfrac{1}{20}x^2 + 5}$.

Team Round

1. Given right $\triangle ABC$ with altitude $\overline{CD}$ in the figure at right, where $AB = 17$, $A_1 =$ area of $\triangle ABC$, $A_2 =$ area of $\triangle CBD$, and $\frac{A_1}{A_2} = \frac{289}{64}$; find A_1.



Solution: Recall the theorem: the altitude to the hypotenuse of a right triangle divides the right triangle into two triangles that are similar to each other and also similar to the original right triangle. In this case, $\triangle ACD \sim \triangle CBD \sim \triangle ABC$. Recall also that the ratio of the areas of two similar triangles is equal to the square of their scale factor. Let f be the scale factor from $\triangle CBD$ to $\triangle ABC$. Then $f^2 = \frac{289}{64}$ and

$$f = \sqrt{\frac{289}{64}} = \frac{\sqrt{289}}{\sqrt{64}} = \frac{17}{8}.$$

Note that $\overline{AB}$ and $\overline{CB}$ are corresponding sides of $\triangle ABC$ and $\triangle CBD$ so that $AB = f \cdot CB$. Then $17 = \frac{17}{8} \cdot CB$, and $CB = \frac{8}{17} \cdot 17 = 8$.

Now AC can be found using the Pythagorian Theorem:

$$\begin{aligned} AB^2 &= AC^2 + CB^2 \\ AC^2 &= AB^2 - CB^2 \\ &= 17^2 - 8^2 = 289 - 64 = 225 \end{aligned}$$

and $AC = \sqrt{225} = 15$. Finally, $A_1 = \frac{1}{2} CB \cdot AC = \frac{1}{2} \cdot 8 \cdot 15 = \frac{8}{2} \cdot 15 = 4 \cdot 15 = \boxed{60}$.

2. Find three consecutive odd integers, in increasing order, such that three times the middle integer is seven more than the sum of the first and last integers.

Solution: Let the middle integer be n so that the sequence is $n-2, n, n+2$. Then

$$3n = n - 2 + n + 2 + 7 = 2n + 7.$$

Subtract $2n$ from both sides of this equation and $3n - 2n = n = 2n - 2n + 7 = 7$, so $n = 7$. The three consecutive integers are therefore $\boxed{5, 7, 9}$. To check, note that $3 \cdot 7 = 21 = 5 + 9 + 7$.

3. Let $f(x) = 7x^2 - 2x + 1$. Find $f(k-1)$. Express your answer as a polynomial in standard form, that is, $ak^2 + bk + c$.

Solution: Substitute $k-1$ for x in $f(x)$, expand $((k-1)^2$, collect terms, and put the result in standard form:

$$\begin{aligned} f(k-1) &= 7\left(k-1\right)^2 - 2(k-1) + 1 \\ &= 7\left(k^2 - 2k + 1\right) - 2k + 2 + 1 \\ &= 7k^2 - 14k + 7 - 2k + 3 \\ &= 7k^2 - (14+2)k + 7 + 3 = \boxed{7k^2 - 16k + 10} \end{aligned}$$

where the identity $(a-b)^2 = a^2 - 2ab + b^2$ was used to expand $(k-1)^2$.

4. Seven cards, exactly one of which is a joker, are randomly placed in a row. Ann is instructed to flip over three of the cards in succession. She is shown the adjacent card or cards to the flipped card after she flips it. Ann chooses a card flipping strategy that will guarantee that she knows the location of the joker before she flips the third card. Find the probability that Ann flips over the joker in her first or second try, given that she flips over the joker on her second flip if it is shown to her after her first flip. Express your answer as a fraction $\frac{m}{n}$.

Solution: To guarantee that Ann knows the location of the joker before flipping the third card, she must see six of the seven cards, either because she flipped them over or because they were shown to her after her first or second flip. Number the seven cards 1, 2, 3, 4, 5, 6, and 7 left to right. One strategy is to turn over card 2, then card 6 if the joker is not 2. Ann will then have seen cards 1, 2, 3, 5, 6, and 7 before her third flip. If the joker is not one of those six cards, then Ann will flip over card 4, which must be the joker. If the joker is card 5 or 7, then Ann will flip it on her third flip.

To simplify the solution, consider the number of cards that have been exposed immediately after the second flip: the first card that is flipped, the two cards adjacent to that card (which were shown to Ann), and the second card that is flipped, for a total of four cards. If the joker is any one of those four cards, then Ann will have flipped over the joker on her first or second try. Thus, the desired probability is $\boxed{\dfrac{4}{7}}$.

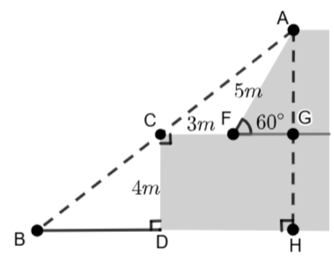
5. Two circles are defined by the equations $(x+6)^2+(y-1)^2=9$ and $(x+1)^2+(y-1)^2=4$. Find all the points of intersection of these circles, expressed as ordered pairs (x,y).

Solution: Observe that the equations of the two circles are each written in standard form:

$$(x-h)^2+(y-k)^2=r^2$$

where the circle is centered at (h,k) with radius r. For this problem, the first equation describes a circle centered at $(-6,1)$ with radius $r_1=\sqrt{9}=3$. The second equation describes a circle centered at $(-1,1)$ with radius $r_2=\sqrt{4}=2$. Note that the distance between the two centers is $\sqrt{(-1-(-6))^2+(1-1)^2}=\sqrt{(-1+6)^2+0^2}=\sqrt{5^2}=5$. In this case, the point $(-6+3,1)=\boxed{(-3,1)}$ is a point of tangency (and the lone point of intersection) of the two circles because it lies on the two circles as well as their common tangent line $(x=-3)$. Note that no other intersection point is possible because it would violate the triangle inequality.

6. A building has the shaded profile shown at right, with a 4m vertical wall $\overline{CD}$, a 3m horizonal surface, and a 5m sloping wall that rises at an angle of 60° from the surface. A ladder is placed on the level and horizontal ground at point B and reaches exactly to point A at the corner of the sloping wall, touching the vertical wall at its corner C. What is the length of the ladder (AB) in m? Express your answer in simplest form as $\frac{m+p\sqrt{q}}{n}$.



Solution: First, construct $\overline{AH}$ where $\overline{AH}\perp\overline{BH}$, the ground. Also label points F and G, as shown in the above figure, where F lies on $\overline{CG}$ and G lies on $\overline{AH}$. Note that $\triangle FGA$ is a $30°-60°-90°$ triangle whose sides are in proportion $1:\sqrt{3}:2$. The side lengths of $\triangle FGA$ are therefore $\frac{5}{2}(1:\sqrt{3}:2)=\frac{5}{2}:\frac{5\sqrt{3}}{2}:5$ so that $FG=\frac{5}{2}$ and $AG=\frac{5\sqrt{3}}{2}$. Note that $CG=CF+FG=3+\frac{5}{2}=\frac{11}{2}$. Apply the Pythagorean Theorem to $\triangle CGA$ to find AC:

$$AC^2=CG^2+AG^2=\left(\frac{11}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2=\frac{11^2}{2^2}+\frac{\left(5\sqrt{3}\right)^2}{2^2}=\frac{121}{4}+\frac{75}{4}=\frac{196}{4}=49$$

and $AC=\sqrt{49}=7$. Next, note that $\triangle CGA\sim\triangle BHA$, and that $AH=AG+GH=AG+CD=\frac{5\sqrt{3}}{2}+4$. Write the proportion between corresponding sides $\frac{AB}{AC}=\frac{AH}{AG}$, solve for AB, and simplify:

$$\frac{AB}{7}=\frac{\frac{5\sqrt{3}}{2}+4}{\frac{5\sqrt{3}}{2}}$$

$$AB=\frac{7\left(\frac{5\sqrt{3}}{2}+4\right)}{\frac{5\sqrt{3}}{2}}\cdot\frac{2}{2}=\frac{7\left(5\sqrt{3}+8\right)}{5\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\frac{35\sqrt{3}^2+56\sqrt{3}}{15}=\boxed{\frac{105+56\sqrt{3}}{15}}$$

7. Let a, b, and c be different integers between 1 and 9, inclusive. Find the largest possible value of $\dfrac{a+b+c}{abc}$.

Solution: Note that if a, b, c take on the largest possible values (7,8,9), then $\dfrac{a+b+c}{abc} = \dfrac{7+8+9}{7\cdot 8\cdot 9} = \dfrac{24}{7\cdot 8\cdot 9} = \dfrac{3}{7\cdot 9} = \dfrac{1}{7\cdot 3} = \dfrac{1}{21}$. On the other hand, if if a, b, c take on the smallest possible values (1,2,3), then $\dfrac{a+b+c}{abc} = \dfrac{1+2+3}{1\cdot 2\cdot 3} = \dfrac{6}{6} = 1$. Note that increasing any single number from $a, b, c = 1, 2, 3$ will cause the product abc to increase more than the sum $a+b+c$, so that the largest value of $\dfrac{a+b+c}{abc}$ is $\boxed{1}$.

8. How many three-digit and four-digit numbers can be formed using the digits 1, 2, 3, 4, 5, and 6, if no digit is repeated within a single number?

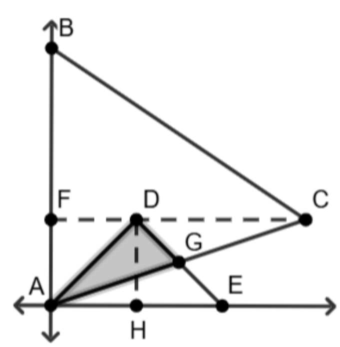
Solution: First count the three-digit numbers. Because digits cannot repeat, there are six choices for the first digit, five choices for the second digit, and four choices for the third digit. In total there are $6\cdot 5\cdot 4 = 120$ three-digit numbers. Next count the four-digit numbers. In this case there will be three choices left for the fourth digit, so there are $6\cdot 5\cdot 4\cdot 3 = 360$ four-digit numbers. In total, there are $120 + 360 = \boxed{480}$ possible three- and four-digit numbers.

9. Given points $A(0,0)$, $B(0,12)$, $C(12,4)$, $D(4,4)$, and $E(8,0)$, find $\frac{A_2 - A_1}{A_2}$, where A_2 is the area of $\triangle ABC$ and A_1 is the area of the overlap between $\triangle ABC$ and $\triangle ADE$.

Solution:

Begin by plotting $A(0,0)$, $B(0,12)$, $C(12,4)$, $D(4,4)$, and $E(8,0)$ and sketching $\triangle ABC$ and $\triangle ADE$. Draw a line parallel to the x-axis through C, Label the point of intersection of this line with the y-axis: $F(0,4)$. $\overline{CF}$ is the altitude drawn to the base $\overline{AB}$ of $\triangle ABC$. It has length 12, as does $\overline{AB}$, so

$$A_2 = \frac{1}{2}\cdot 12\cdot 12 = 6\cdot 12 = 72.$$



To find A_1, find the intersection of $\overline{AC}$ with $\overline{DE}$ and label it G. Note that G has coordinates $(6,2)$ and AG is a median of $\triangle ADE$ (because $DG = 2\sqrt(2) = GE$). Recall that the median of a triangle divides the triangle into two triangles of equal areas. Therefore A_2 is equal to half of the area of $\triangle ADE$. The altitude $\overline{DH}$ drawn from D to the x-axis is four units long, and $AE = 8$. Therefore the area of $\triangle ADE$ is $\frac{1}{2}\cdot 4\cdot 8 = 16$. Finally, $A_1 = \dfrac{1}{2}\cdot 16 = 8$, and $\dfrac{A_2 - A_1}{A_2} = \dfrac{72-8}{72} = \dfrac{64}{72} = \boxed{\dfrac{8}{9}}$.